Minimal Model Program Learning Seminar.

Week 22:

- Fano type varieties.
- Rationally connected varieties.

Rationally connected varieties:

(X,Δ) klt singularities.

(X,Δ) is of log general type if $K_X+\Delta$ is big
(X,Δ) is log Calabi–Yau if $K_X+\Delta \equiv 0$.
(X,Δ) is log Fano if $-(K_X+\Delta)$ is ample

} positivity of ω_X.

Can we "classify" by looking at rational curves:

Uniruled: through a general point $x \in X$ there is a $\mathbb{P}^1$.

$\Uparrow$

Rationally connected: For any $x,y \in X$ general, there is a $\mathbb{P}^1$ passing through x & y.

$\Downarrow$

RCC: For any $x,y \in X$ general, there is a connected chain of $\mathbb{P}^1$'s passing through x & y

unirational: X of dim n admits $\mathbb{P}^n \dashrightarrow X$ rational generically finite

$\Uparrow$

rational: X of dim n is rational $\mathbb{P}^n \dashrightarrow X$ birational

rational

$\Downarrow$

unirational.

$\Downarrow$??

rationally connected

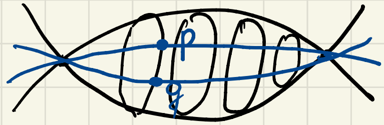
$\Downarrow$

rationally chain connected

$\Downarrow$

uniruled.

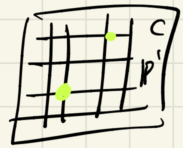
= existence of examples that $\Uparrow$ does not hold.



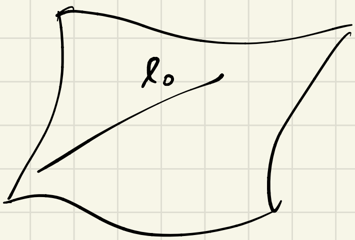
Cone over elliptic curve is RCC but not RC

$C \times \mathbb{P}^1$

$g \geq 1$



Unirational but not rational: X cubic 3-fold, $X \subseteq \mathbb{P}^4$



$X \subseteq \mathbb{P}^4$

$W = \{(p, L) \mid p$ is in L_0 and L is tangent to X at $p\}$.

W is a $\mathbb{P}^2$-bundle over L_0.

$\mathbb{P}^3$

is bir

W is rational.

$\varphi : W \dashrightarrow X$, "maps (p, L) to the third intersection point of L with X"

2 : 1.

X is not rational: Middle Hodge structure of X is not the Jacobian of a curve.

How these approaches compare?

- X uniruled "$\Longrightarrow$" ω_X is negative.

what happens if ω_X is positive or ω_X trivial?

- X smooth with no rational curves $\Longrightarrow$ ω_X is nef.
- $K_X \sim_{\mathbb{Q}} 0$, there could be no rat curves (Ab var)

  dense set of rat curves (K3 surface)
- If X is uniruled, then K_X is not pseudo-effective.

Remark: X rationally connected $\sim$ "ω_X negative".

RC varieties are block-wise Fano type varieties.

Theorem (Graber - Harris - Mazur - Starr 2001): $X \longrightarrow B$

s.t. B & general fiber are RC $\Longrightarrow$ X is RC.

Proposition: Image of RC is again RC

Warning: This is a big flash-forward. $\boxed{\text{Fano} \Rightarrow \text{RC.}}$

RC $\quad$ RC $\quad$ 2-ray game.

$$\begin{tikzcd}
X \arrow[r, dashed, "K_X\text{-MMP}"] & X_1 \arrow[r, dashed] \arrow[d] & X_2 \arrow[r, dashed] \arrow[d] & X_3 \arrow[d] \\
& Z_1 \arrow[r, dashed, "K_{Z_1}\text{-MMP}"] & Z_2 \arrow[d] & Z_3 \arrow[d] \\
& & W_2 & W_3 \arrow[d] \\
& & & Y_3
\end{tikzcd}$$

general fiber is Fano

Z_2 RC

Z_1 RC

general fiber is Fano

K_{Z_1} not psef

W_2 RC

RC

$$\begin{tikzcd}
X \arrow[r, dashed] & \underbrace{M \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \cdots \arrow[r] & M_n}_{\text{tower of MFS.}} & Y_3
\end{tikzcd}$$

Fano fibers

RC are "block-wise" Fano varieties:

$$\{\text{Fano varieties}\} \subseteq \{\text{RC varieties}\} \sim \left\{\begin{matrix}\text{Towers of} \\ \text{MFS}\end{matrix}\right\}$$

birational to.

Ex: Bott or gen Bott towers.

Topology: $X \subseteq \mathbb{P}^N$ smooth proj over $\mathbb{C}$.

$$\pi_1^{top}(C) \twoheadrightarrow \pi_1^{top}(X)$$

C is a complete int of hyperplanes inside X.

Theorem (Kollár): X RC, for every $x \in X$,

smooth projective

there is a family of rational curves. $F: W \times \mathbb{P}^1 \longrightarrow X$

$$F(W \times (0:1)) = \{x\}$$

for every $w \in W$, we have

$$\pi_1(F^{-1}(X), (w, (0:1))) \longrightarrow \pi_1(X, x) \quad \text{surjective}$$

In particular, a smooth proper RC variety is simply connected.

Can a RC variety be a universal cover? X smooth

X RC then it has no holomorphic forms,

By Hodge theory, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

In particular $\chi(\mathcal{O}_X) = 1$.

$$\begin{array}{ccc} X & \xleftarrow[f]{\text{étale}} & Y \\ \cup & & \cup \\ \mathbb{P}^1 & \longleftarrow & \mathbb{P}^1 \end{array}$$

, Y will be also RC.

Then $\chi(\mathcal{O}_Y) = 1$

$$\chi(\mathcal{O}_Y) = \deg f \, \chi(\mathcal{O}_X). \implies \deg f = 1$$

Theorem (Kontsevich - Tschinkel 2017):

$\pi : \mathcal{X} \longrightarrow B$ $\qquad$ $\pi' : \mathcal{X}' \longrightarrow B$

smooth proper morphisms. B smooth connected curve over char 0.

If the generic points of π and π' are birational over $\mathbb{K}(B)$.

Then for any point $b \in B$, the fibers X_b and $X'_b = \pi'^{-1}(b)$,

are birational over the residue field of b.

In particular, if the generic fiber is rat (resp RC),

then every fiber is rat (resp. RC).

Maximally RC fibration:

X equivalence $(x,y) \in R \iff$ x and y can be connected by rational curves.

Q: $X \xrightarrow{\phi} Y$ so that the fibers are the equivalence classes of R.

Theorem: X smooth complex projective varieties.

There exists $X^\circ \subseteq X$ open, a normal variety T°, and a proj surj morphism $X^\circ \xrightarrow{\varphi^\circ} T^\circ$ s.t.:

1.- The general fiber of φ° is RC.

2.- the very general fiber of φ° is an equivalence class of $\mathcal{R}$

Moreover, this morphism is unique up to $\sim_{bir}$.

Example: X RCC, $\varphi: X \longrightarrow \mathrm{Spec}(\mathbb{C})$.

X K3 surface, $\varphi: X \longrightarrow X$.

X v.b over Ab, $\varphi: X \longrightarrow Ab$.

$$\begin{tikzcd} X \arrow[d] & X^{ter} \arrow[l] \arrow[d] \\ Y & Y^{ter} \arrow[l] \end{tikzcd}$$

neg coeff in the boundary

$\underbrace{K_X \text{ is pseff} + \text{RC.}}_{Y \text{ not terminal}}$

Q: For which kind of varieties can we realized the MRC fibration MMP-wise

Theorem: Let (X, Δ) be a log pair and let $f : X \longrightarrow S$ be a projective birational morphism such that $-K_X$ is relatively big and $\mathcal{O}_X(-m(K_X + \Delta))$ is relatively generated for some $m > 0$.

Let $g : Y \longrightarrow X$ be any birational morphism and let $\pi : Y \longrightarrow X$ be the composition morphism.

Then, every fiber of π is rationally chain connected modulo the inverse image of non-klt (X, Δ).

Birational morphism ~ composition of FT morphisms.

$\varphi : X \longrightarrow Y$. birational, both X & Y have term sing

$\mathbb{Q}$-fact

$\varphi^*(K_Y) = K_X - \sum' a_i E_i$, $a_i > 0$.

There exists

$E \subseteq X$ effective supported on $Ex(\varphi)$ with $-E$ ample over Y.

A ample on X, $\underline{\varphi^* \varphi_* A} = A + E$, $A + E \sim_{\mathbb{Q},X} 0$.

Understand $B_{s-}(K_X)$ over Y.

$$B_{s-}(K_X) = \bigcup_{\varepsilon > 0} B_s(K_X - \varepsilon E) \supseteq Ex(\varphi)$$

$K_X - \varepsilon E \sim_{\mathbb{Q},X} F \geq 0$, then

$\sum' a_i E_i - \varepsilon E \sim_{\mathbb{Q},X} F$, so

$$F \underbrace{- \sum' a_i E_i + \varepsilon E} \sim_{\mathbb{Q},X} 0.$$

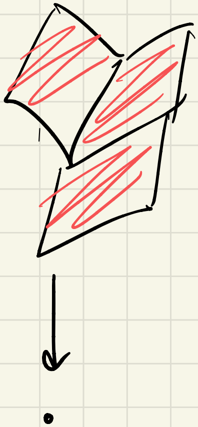
$$F - \sum' a_i' E_i \sim_{\mathbb{Q},X} 0.$$

$\varphi_* F$ is eff, neg Lemma $F - \sum' a_i' E_i \geq 0$

$F \geq \sum' a_i' E_i$

$B_{s-}(K_X) \supseteq Ex(\varphi)$.

$$X \longrightarrow Y$$



K_X − MMP over Y.

every step is a FT morphism

Fano type morphism: $X \longrightarrow Z$ is said to be Fano type if there exists $\Delta \geq 0$ on X:

i) (X, Δ) klt sing,

ii) $-(K_X + \Delta)$ is nef & big over Z.

$\mathbb{Q}$-fact

3-fold

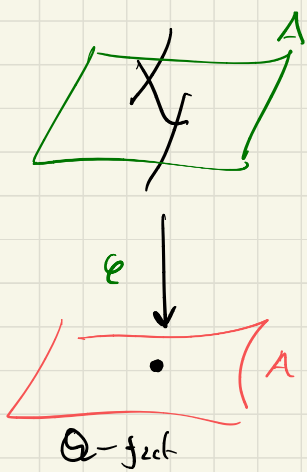
$$\begin{tikzcd} X \arrow[r, dashed, "K_X\text{-MMP}"] \arrow[d] & X' \arrow[dl, "\mathrm{id}"'] \\ Y & \end{tikzcd}$$

Y

X' $\mathbb{Q}$-fact.

← small morphism

Y $\mathbb{Q}$-fact.

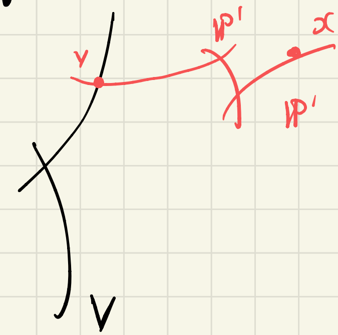


$$\varphi^*(\varphi_* A) = A \implies A \sim_{\mathbb{Q}, X} 0.$$

Cor 1: (X, Δ) log pair, such that $-(K_X+\Delta)$ semiample + big. Then

$$\pi_1(\text{Non-klt}(X,\Delta)) \twoheadrightarrow \pi_1(X).$$

Proof:



Assumption $V \neq \emptyset$.

Campana's Theorem.

$\left\{ \begin{array}{l} \pi_1(V) \text{ in } \pi_1(X) \\ \text{has finite index.} \end{array} \right.$

$$\begin{array}{c} X \\ \uparrow f \\ Y \end{array}$$

f be an étale finite cover.

$W = \text{nonklt}(Y, \Gamma)$

$\Gamma = f^*\Delta$.

$-(K_Y+\Gamma)$ semiample + big.

In a theorem in Kollár–Mori

$\text{non-klt}(Y,\Gamma) = \pi^{-1}(V) = W$

is connected.

Assume $V = \emptyset$.

(X, Δ) is klt. $\quad Y \xrightarrow[f]{\text{étale}} X$

By KV vanishing $h^i(Y, \mathcal{O}_Y) = h^i(X, \mathcal{O}_X) = 0$.

$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 1$ so f has degree 1

Then $\pi_1(X) = \{1\}$

Lemma: Let (X, Δ) klt, $f: X \longrightarrow S$ proj morphism

Suppose $-(K_X + \Delta)$ is nef over S and $-K_X$ is rel big over S. Then $-(K_X + \Delta)$ is semiample over S.

Lemma: Let (X, Δ) klt, $f: X \longrightarrow S$ proj morphism

Suppose $-(K_X+\Delta)$ is nef over S and $-K_X$ is rel big over S. Then $-(K_X+\Delta)$ is semiample over S.

Proof:

Claim: There exists $\Theta \geq 0$ for which (X, Θ) ✓ is klt and $-(K_X+\Theta)$ is ample. ✓

proof of the claim: $-K_X \sim_{\mathbb{Q},S} A+B$

$A \longrightarrow$ ample over S

$B \longrightarrow$ effective

$\Theta = (1-\varepsilon)\Delta + \varepsilon B$. Then

$$-(K_X+\Theta) \sim_{\mathbb{Q},S} \underbrace{\underbrace{-(1-\varepsilon)(K_X+\Delta)}_{\text{nef over } S} + \underbrace{\varepsilon A}_{\text{ample over } S}}_{\text{ample over } S}.$$

For $\varepsilon \ll 1$ the pair (X, Θ) remains klt. □

D nef, $D-(K_X+\Theta)$ ample.

bpf $\Longrightarrow$ $|mD|$ base point free over S. □

Corollary 2: (X,Δ) klt, $f: X \longrightarrow S$ proj.
$-K_X$ is big over S & $-(K_X+\Delta)$ nef over S.
Then every fiber is RCC.

Proof: Trivial after the Lemma + Theorem.

Corollary 3: Let (X,Δ) be a dlt pair.
If $g: Y \longrightarrow X$ is birational, then the fibers are RCC.

Proof: Kollár-Mori: dlt is approximation of klt.
Hence, the statement follows from the theorem.

Corollary 2: (X,Δ) klt, $f: X \longrightarrow S$ proj.
$-K_X$ is big over S & $-(K_X+\Delta)$ nef over S.
Then every fiber is RCC.

Remark: FT $\Longrightarrow$ RCC. $\xRightarrow{+\text{klt}}$ RC.

X FT, (X,Δ) klt and $-(K_X+\Delta)$ big & nef

$X \longrightarrow \operatorname{Spec}(\mathbb{K})$, $-K_X = \underbrace{\underbrace{-(K_X+\Delta)}_{\text{big}} + \underbrace{\Delta}_{\text{eff}}}_{\text{big}}$

X is RCC

Corollary 4: (X,Δ) dlt, X RCC $\Longleftrightarrow$ X RC.

Corollary 4: (X,Δ) dlt, X RCC $\iff$ X RC

Proof:

$$\begin{array}{c} Y \\ \Big\downarrow \varphi \\ (X,\Delta) \\ \text{RCC} \end{array}$$

Y smooth

fibers of φ are RCC.

X RCC + fibers RCC

$\Downarrow$

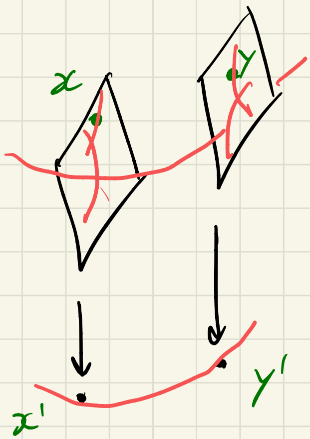
Y is RCC

$\Downarrow$ Y smooth

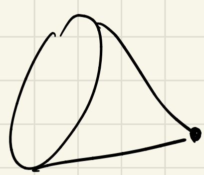
Y is RC

$\Downarrow$

X is RC.



Remark:



Cone over elliptic curve

is lc is not dlt

is RCC but not RC